MAX PLANCK INSTITUTE FOR INTELLIGENT SYSTEMS

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MODEL TRANSPORT: TOWARDS SCALABLE TRANSFER LEARNING ON MANIFOLDS

SUPPLEMENTAL MATERIAL

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Abstract. This technical report is complementary to [1] and contains proofs, explanation of the attached video (visualization of bases from the body shape experiments), and high-resolution images of select results of individual reconstructions from the shape experiments. It is identical to the supplemental material submitted to the Conference on Computer Vision and Pattern Recognition (CVPR 2014) on November 2013.

References

[1] Freifeld, O., Hauberg S., and Black, M.J.: Model Transport: Towards Scalable Transfer Learning on Manifolds. CVPR (2014)

Supplemental Material

Model Transport:
Towards Scalable Transfer Learning on Manifolds

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Contents

1. Proofs

In this section we provide the proofs for the propositions from the paper. Along the way, we also prove a fact mentioned in the paper – that every metric parallel parallel transport is an invertible linear map.

Recall, in the notation from the paper, that parallel transport satisfies:

(I) $\Gamma^c_{s,s}$ is the identity map on $T_{c(s)}M$;

(II) $\Gamma^c_{u,t} \circ \Gamma^c_{s,u} = \Gamma^c_{s,t}$;

(III) $\Gamma^c_{s,t}$ depends smoothly on s and t.

Claim 1. *Parallel transport between tangent spaces is bijective.*

Proof. By (I), $\Gamma^c_{t,s} \circ \Gamma^c_{s,t} = \Gamma^c_{s,s}$. By (II) $\Gamma^c_{s,s}$ is the identity map on $T_{c(s)}M$. Thus $\Gamma^c_{s,t}$ is a bijection between $T_{c(s)}M$ and a subset of $T_{c(t)}M$. We need to show that this subset is in fact the whole of $T_{c(t)}M$. Similar reasoning shows that $\Gamma^c_{t,s}$ is a bijection between $T_{c(t)}M$ and a subset of $T_{c(s)}M$. We now use a proof by contradiction. Suppose there exists y in $T_{c(t)}M$ such that $y \notin \Gamma^c_{s,t}(T_{c(s)}M)$. We know that $\Gamma^c_{t,s}(y) \in T_{c(s)}M$ and thus $z \triangleq \Gamma^c_{s,t}(\Gamma^c_{t,s}(y))$ is in the image of $\Gamma^c_{s,t}$. By assumption, $z \neq y$. However, this is in contradiction to the fact that $\Gamma^c_{t,s} \circ \Gamma^c_{s,t}$ is the identity map on $T_{c(t)}M$. □

Of course, bijectivity implies surjectivity. The following theorem is well known from functional analysis.

Theorem (Mazur-Ulam)**.** *Every surjective isometry between two normed linear spaces is affine.*

Here, the term *isometry* means a distance-preserving map.

Claim 2. *Every surjective inner-product-preserving map between two inner-product spaces is linear.*

Proof. An inner-product-preserving map between two inner-product spaces is an isometry. If the map is also surjective then by the Mazur-Ulam Theorem it is affine. Let $f : U \to V$ be a surjective inner-product-preserving map between two inner-product spaces U and V. Thus, f is affine: $f : x \mapsto Ax + b$ where A is linear map $A : U \to V$ and $b \in V$. We need to show that $b = 0_V$, the zero element of V. If $x, y \in U$ then,

$$\langle x, y\rangle_U = \langle Ax + b, Ay + b\rangle_V = \langle Ax, Ay\rangle_V + \langle Ax, b\rangle_V + \langle b, Ay\rangle_V + \|b\|^2_V \ . \quad (1)$$

Taking $x = y = 0_U$, we get that

$$\begin{aligned} 0 &= \|0_U\|_U^2 \\ &= \|A0_U\|_V^2 + \langle A0_U, b\rangle_V + \langle b, A0_U\rangle_V + \|b\|_V^2 \\ &= \|0_V\|_V^2 + \langle 0_V, b\rangle_V + \langle b, 0_V\rangle_V + \|b\|_V^2 \\ &= 0 + 0 + 0 + \|b\|_V^2 \ , \end{aligned} \tag{2}$$

implying that $b = 0_V$. □

The following corollary follows immediately.

Corollary 1. *Every metric parallel transport is linear.*

We now prove Proposition 4.1 (Covariance/PCA Transport) from the paper.

Proof. Recall that VS^2V^T is the eigen-decomposition of $XX^T \in \mathbb{R}^{n\times n}$ and VSU^T is the SVD of $X \in \mathbb{R}^{n\times N}$. Let $A \triangleq V^TX = SU^T$, $A \in \mathbb{R}^{n\times N}$. Denote the columns of A by $A = [a_1, \ldots, a_N] = [Su_1^T, \ldots, Su_1^T]$ where $[u_1, \ldots, u_1] \triangleq U$. Thus, $X = VV^TX = VA \in (T_pM)^N \cong \mathbb{R}^{n\times N}$ and so $x_j = Va_j = \sum_{k=1}^n v_k A_{k,j} \in T_pM \cong \mathbb{R}^n$. Applying a metric parallel transport, we get (for $\tilde{x}_j \in T_qM \cong \mathbb{R}^n$),

$$\tilde{x}_j = \widetilde{\sum_{k=1}^n v_k A_{k,j}} \overset{\text{linearity}}{=} \sum_{k=1}^n \tilde{v}_k A_{k,j} = \tilde{V}a_j \ . \tag{3}$$

In other words, $\tilde{X} = \tilde{V}A = \tilde{V}SU^T$. Since V is orthogonal and inner products are preserved, we know that $\tilde{V}$ is orthogonal too. And since S is diagonal and U is orthogonal, it follows that $\tilde{V}SU$ is indeed the SVD of $\tilde{X}$; namely, the left-singular vectors of $\tilde{X}$ are exactly the transported left-singular vectors of X, while the singular values and right-singular vectors are unchanged. As an aside remark, note that $A \overset{\text{by def.}}{=} V^TX = \tilde{V}^T\tilde{X}$ since inner products are preserved. The fact that $\tilde{V}S^2\tilde{V}^T$ is the eigen-decomposition of $\tilde{X}\tilde{X}^T \in \mathbb{R}^{n\times n}$ follows from $\tilde{X}\tilde{X}^T = \tilde{V}SUU^TS\tilde{V}^T = \tilde{V}S^2\tilde{V}^T$, concluding the proof of part (i) of the proposition. Finally, by the usual connections between SVD and PCA, part(ii) is a direct consequence of part (i). □

We now prove Proposition 4.2 (Simple-Linear-Regression Transport) from the paper.

Proof.

$$\beta = \arg\min_{\alpha\in T_pM} \sum_{i=1}^N l_i(x_i^T\alpha + \beta_0) \tag{4}$$

$$= \arg\min_{\alpha\in T_pM} \sum_{i=1}^N l_i(\langle x_i, A_p^{-1}\alpha\rangle_p + \beta_0) \tag{5}$$

$$= \arg\min_{\alpha\in T_pM} \sum_{i=1}^N l_i(\langle Lx_i, LA_p^{-1}\alpha\rangle_q + \beta_0) \tag{6}$$

$$= \arg\min_{\alpha\in T_pM} \sum_{i=1}^N l_i(\langle Lx_i, A_q^{-1}A_qLA_p^{-1}\alpha\rangle_q + \beta_0) \ . \tag{7}$$

In the third equality we used the fact that L preserves inner products. The result above, together with the fact that

$$\left\{A_qLA_p^{-1}\alpha : \alpha \in T_pM\right\} \tag{8}$$

coincides with T_qM (the map $T_pM \to T_qM : x \mapsto A_q L A_p^{-1}$ is bijective), implies that

$$\begin{aligned} A_q L A_p^{-1}\beta &= \arg\min_{\delta \in T_qM} \sum_{i=1}^{N} l_i(\langle Lx_i, A_q^{-1}\delta\rangle_q + \beta_0) \\ &= \arg\min_{\delta \in T_qM} \sum_{i=1}^{N} l_i((Lx_i)^T\delta + \beta_0) \ . \end{aligned} \tag{9}$$

□

2. Visualizing bases for the body shape experiments

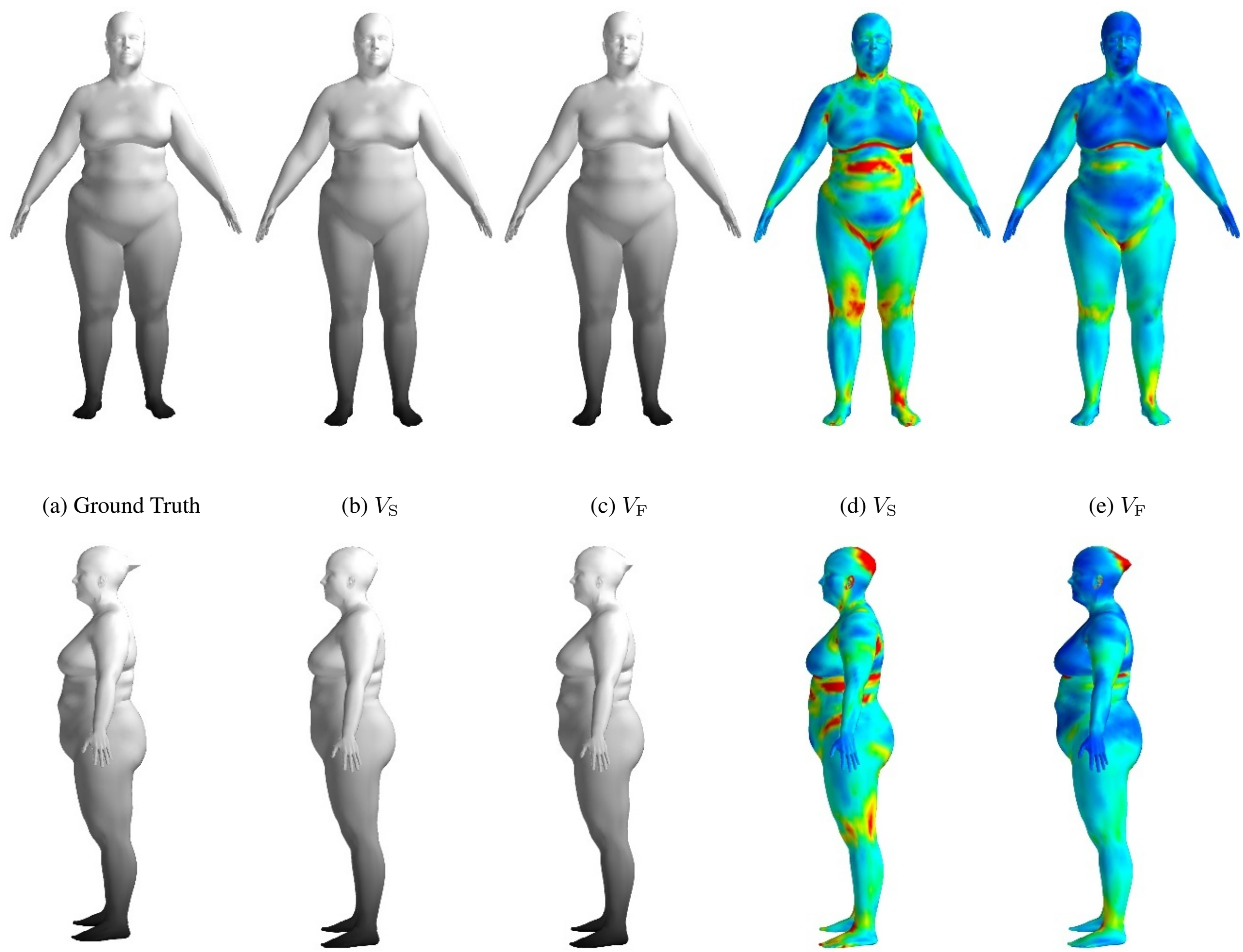
Please see the *video* in the following link:

`http://people.csail.mit.edu/freifeld/ModelTransport/EigenVecs_and_their_PT.mp4`

In the video we show synthesis results as we move along the first few principal components. The first sequence in the movie is for the gender experiment, while the second sequence is for the BMI experiment. On the left, we show the original V_{S} basis while on the right we show its parallel transport, V_Γ. For example, in the first sequence, note how feminine shape changes (left) look upon their parallel transport to the tangent space at the mean man shape (right).

3. Additional Results from the Shape Experiments Described in the Paper

See figures in the next pages. We first show results for 10 test examples from the BMI experiment, and then another 10 from the gender experiment. Each figure shows, from left to right: Ground Truth; V_{S} reconstruction; V_{F} reconstruction; V_{S} reconstruction error; V_{F} reconstruction error. Frontal view is shown in the top row and profile is shown in the bottom. Typically differences in reconstruction results are most noticeable in regions where the corresponding error maps differ significantly in colors.

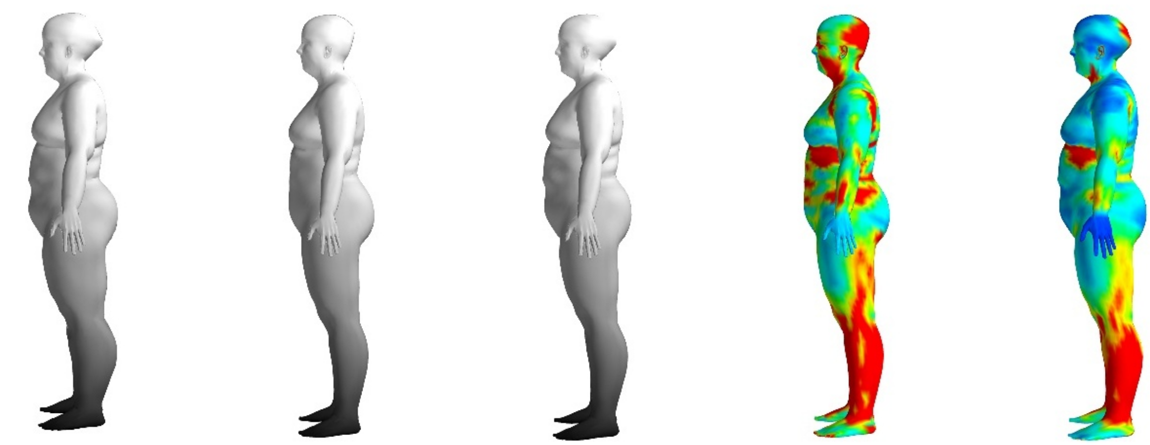


(a) Ground Truth (b) V_{S} (c) V_{F} (d) V_{S} (e) V_{F}

(f) Ground Truth (g) V_{S} (h) V_{F} (i) V_{S} (j) V_{F}

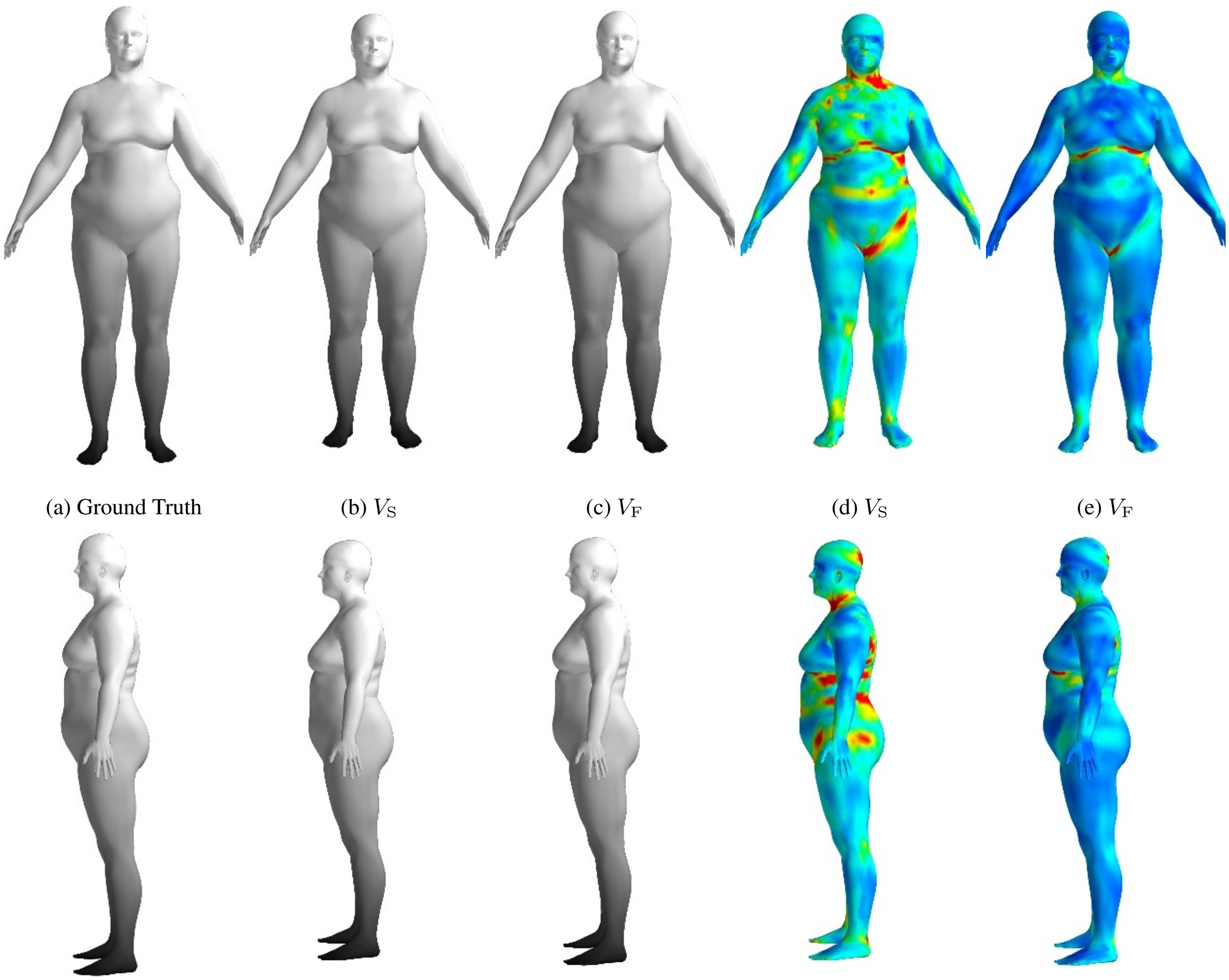
Figure 1: BMI reconstruction example.

(a) Ground Truth (b) V_{S} (c) V_{F} (d) V_{S} (e) V_{F}



(f) Ground Truth (g) V_{S} (h) V_{F} (i) V_{S} (j) V_{F}

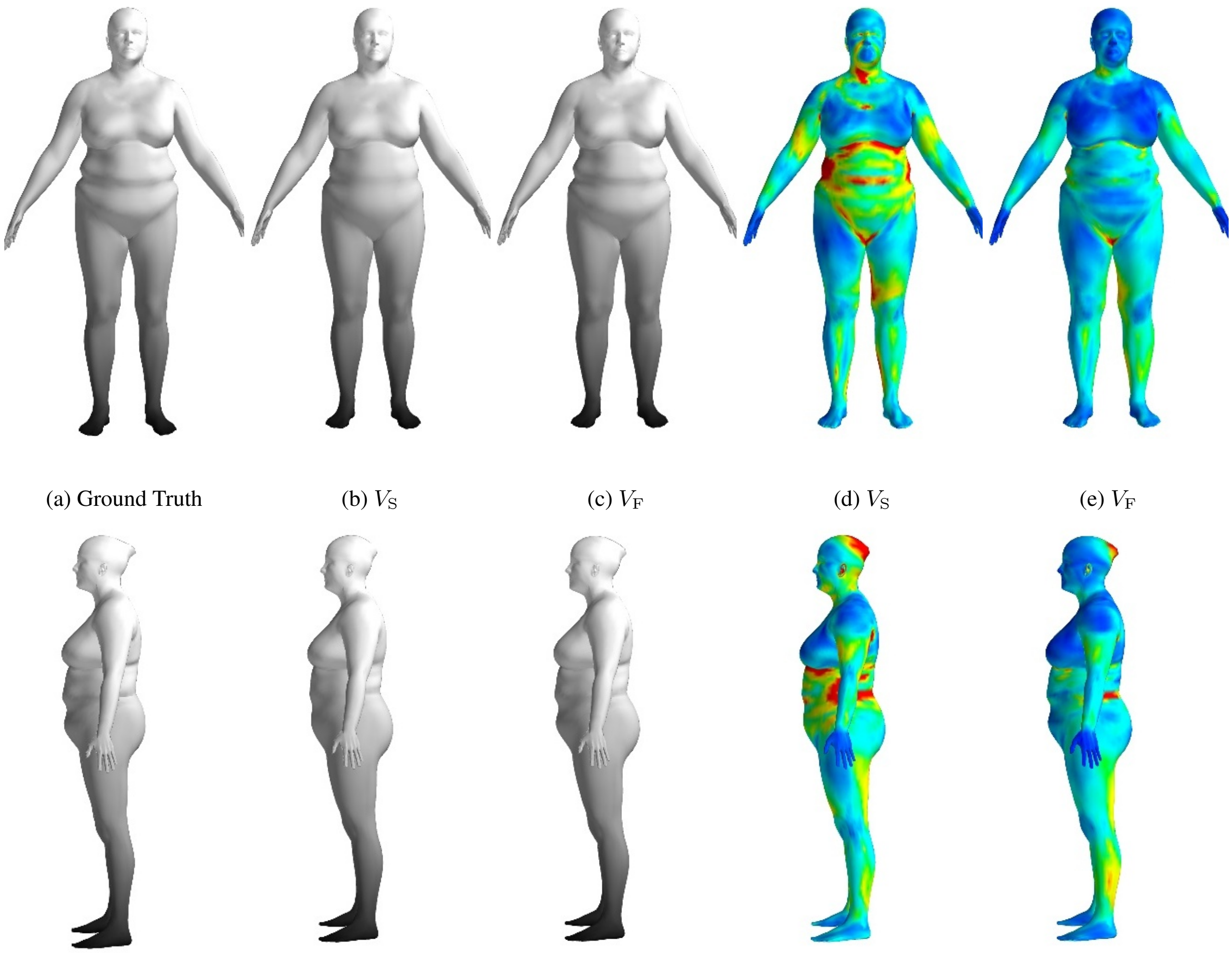
Figure 2: BMI reconstruction example.



(a) Ground Truth (b) V_{S} (c) V_{F} (d) V_{S} (e) V_{F}

(f) Ground Truth (g) V_{S} (h) V_{F} (i) V_{S} (j) V_{F}

Figure 3: BMI reconstruction example.

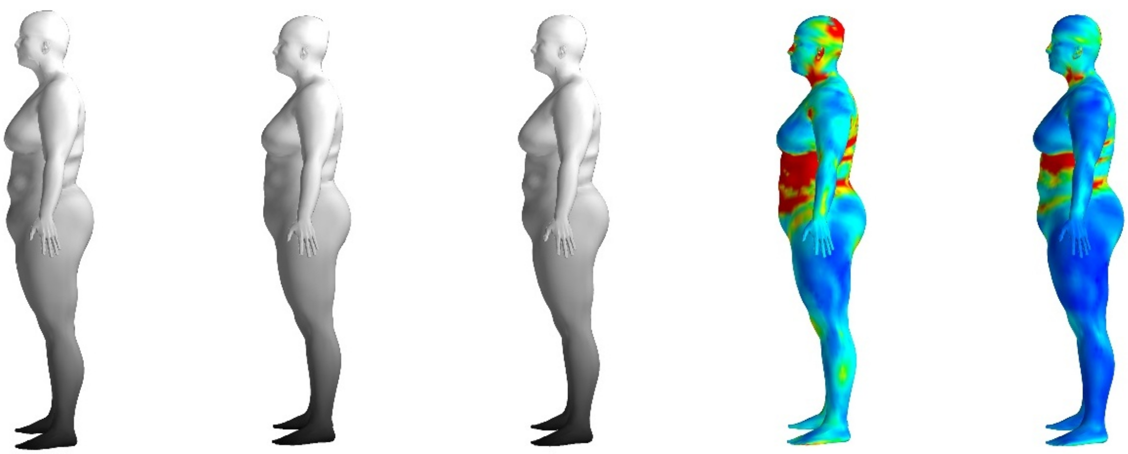


(a) Ground Truth (b) V_{S} (c) V_{F} (d) V_{S} (e) V_{F}

(f) Ground Truth (g) V_{S} (h) V_{F} (i) V_{S} (j) V_{F}

Figure 4: BMI reconstruction example.

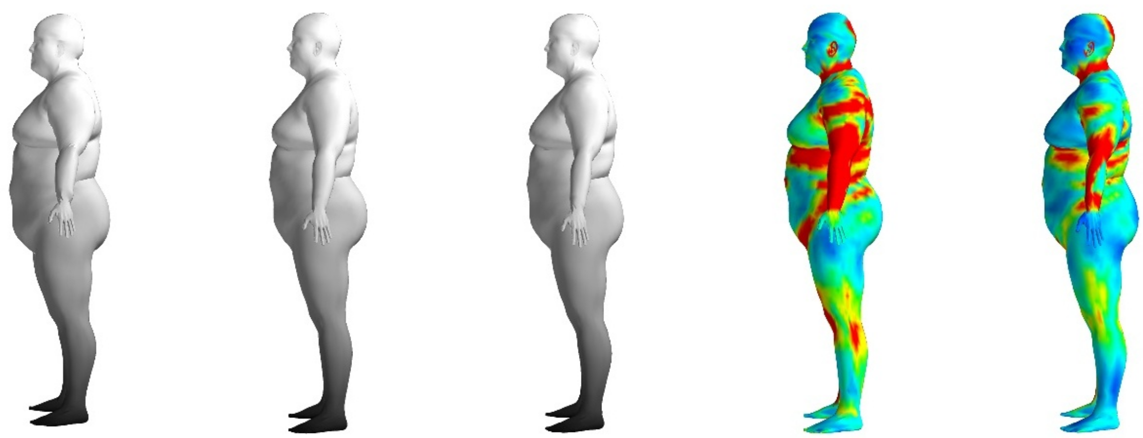
(a) Ground Truth (b) V_{S} (c) V_{F} (d) V_{S} (e) V_{F}



(f) Ground Truth (g) V_{S} (h) V_{F} (i) V_{S} (j) V_{F}

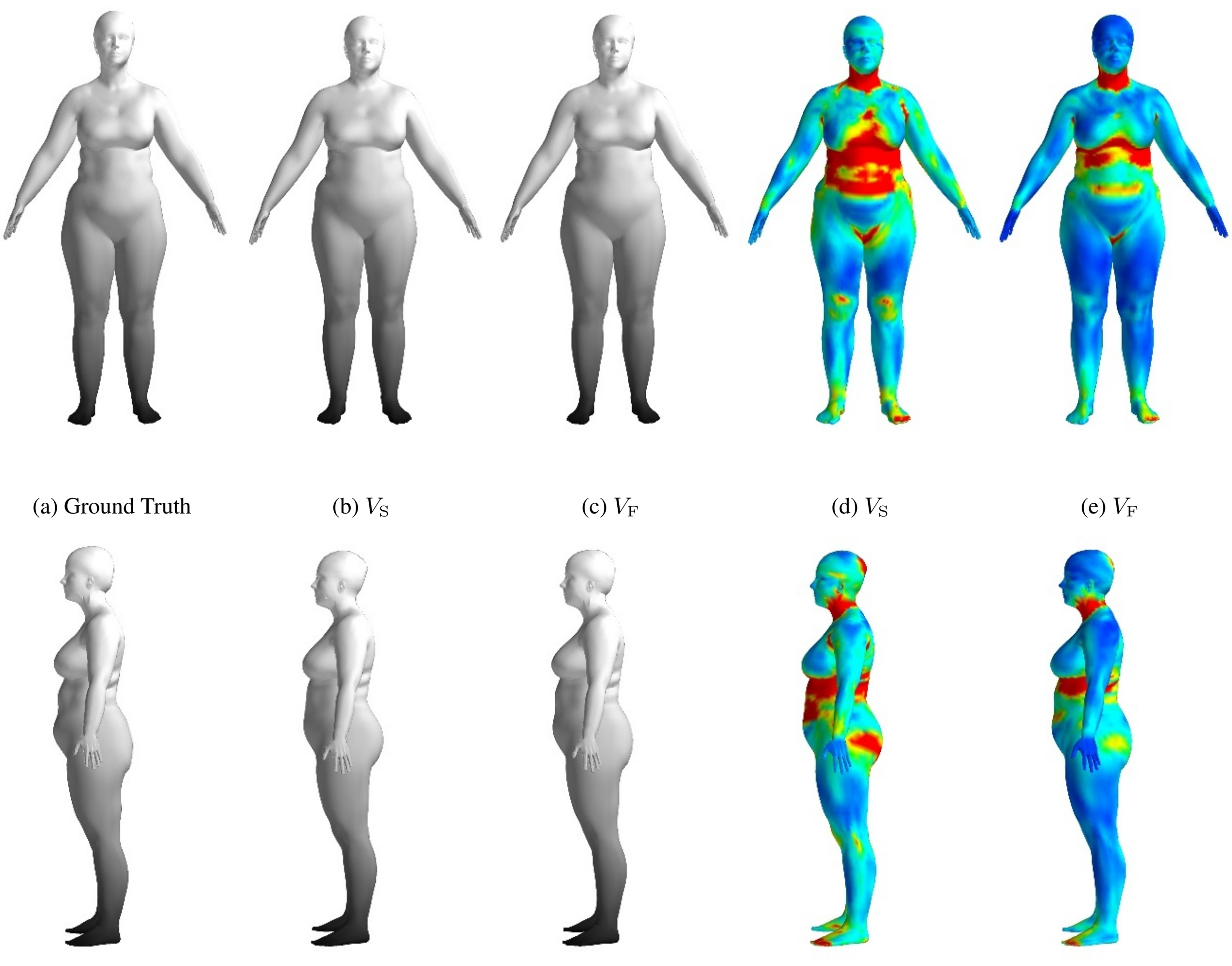
Figure 5: BMI reconstruction example.

(a) Ground Truth (b) V_{S} (c) V_{F} (d) V_{S} (e) V_{F}



(f) Ground Truth (g) V_{S} (h) V_{F} (i) V_{S} (j) V_{F}

Figure 6: BMI reconstruction example.

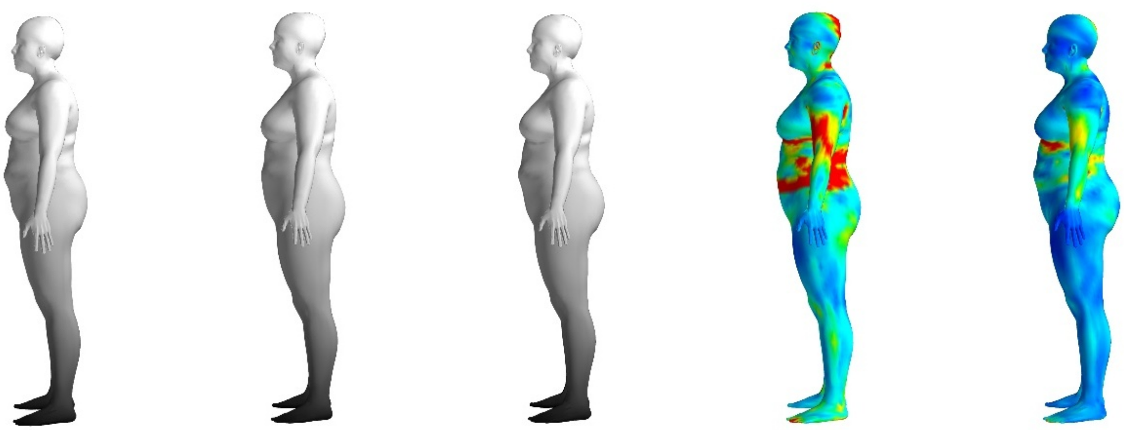


(a) Ground Truth (b) V_{S} (c) V_{F} (d) V_{S} (e) V_{F}

(f) Ground Truth (g) V_{S} (h) V_{F} (i) V_{S} (j) V_{F}

Figure 7: BMI reconstruction example.

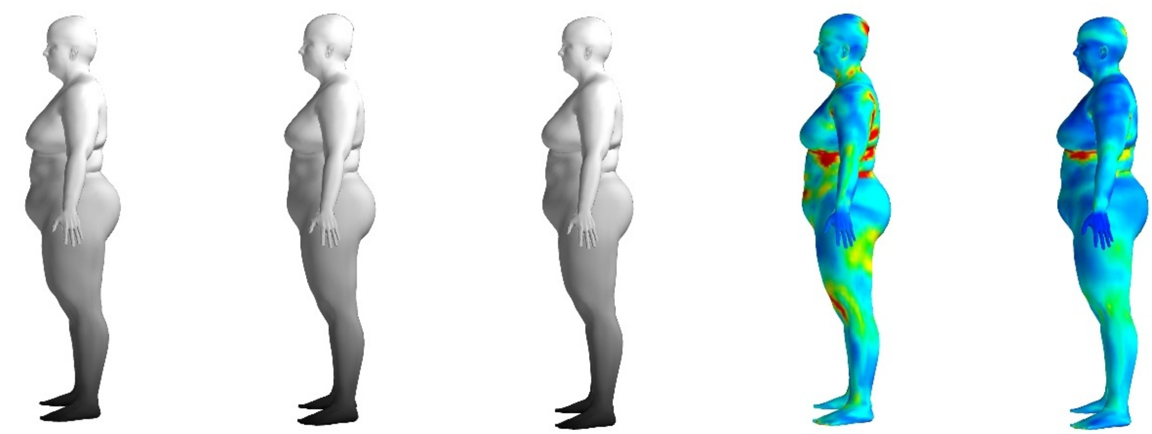
(a) Ground Truth (b) V_{S} (c) V_{F} (d) V_{S} (e) V_{F}



(f) Ground Truth (g) V_{S} (h) V_{F} (i) V_{S} (j) V_{F}

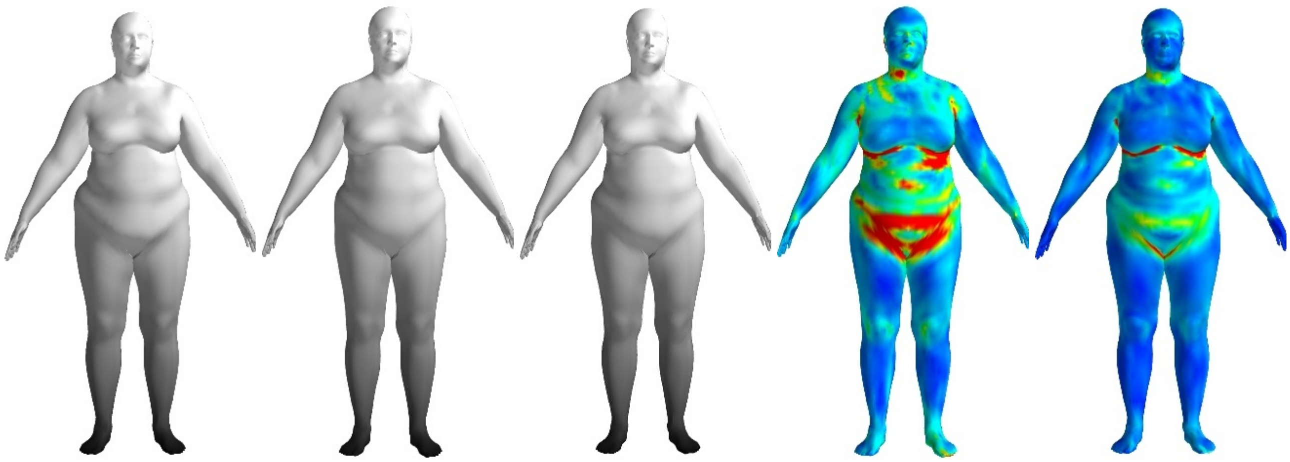
Figure 8: BMI reconstruction example.

(a) Ground Truth (b) V_{S} (c) V_{F} (d) V_{S} (e) V_{F}

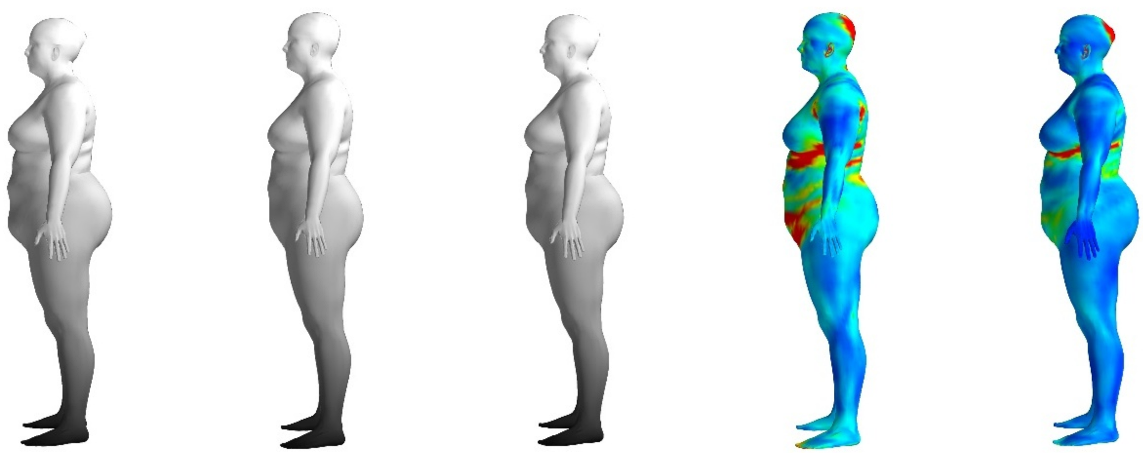


(f) Ground Truth (g) V_{S} (h) V_{F} (i) V_{S} (j) V_{F}

Figure 9: BMI reconstruction example.

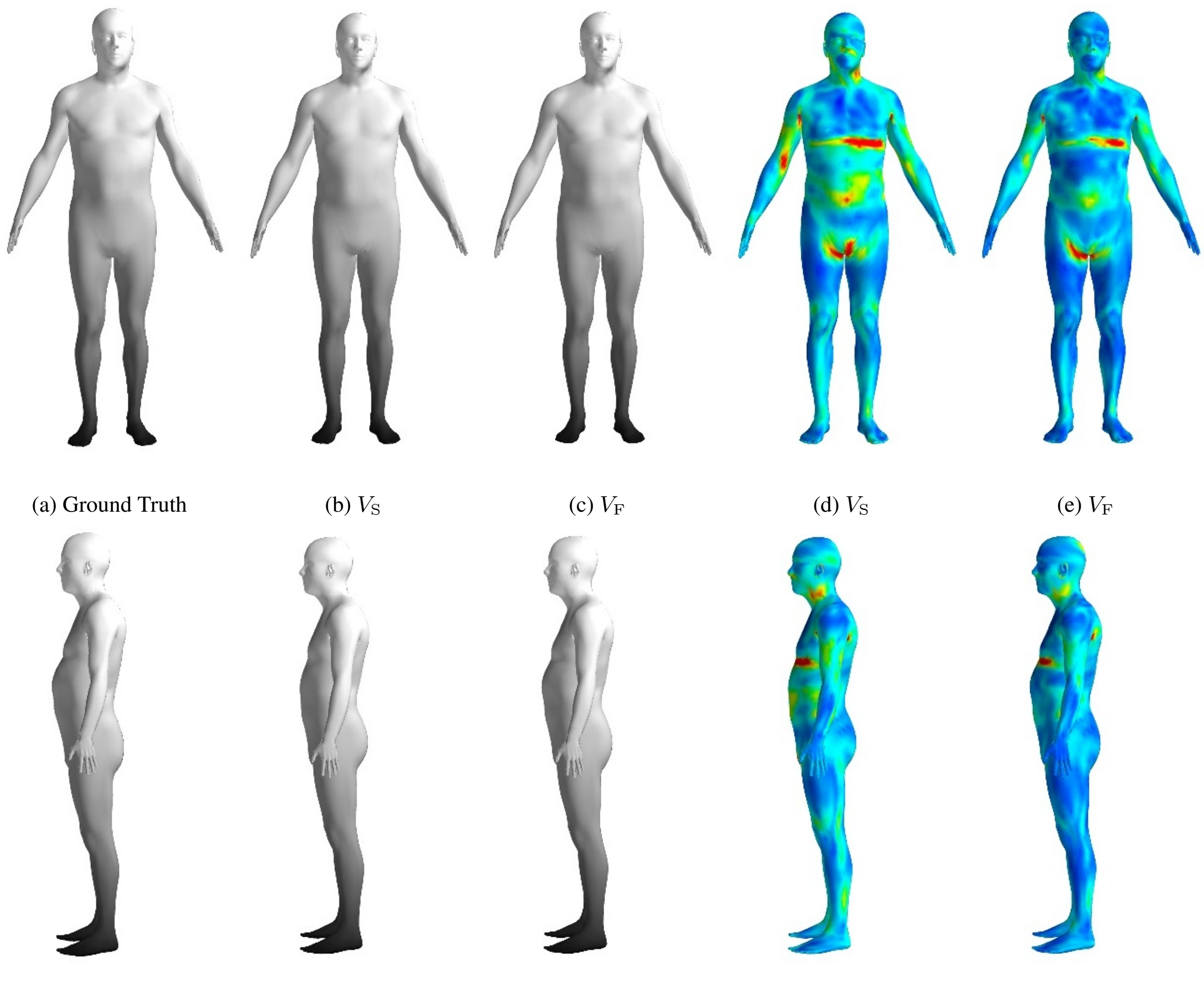


(a) Ground Truth (b) V_{S} (c) V_{F} (d) V_{S} (e) V_{F}



(f) Ground Truth (g) V_{S} (h) V_{F} (i) V_{S} (j) V_{F}

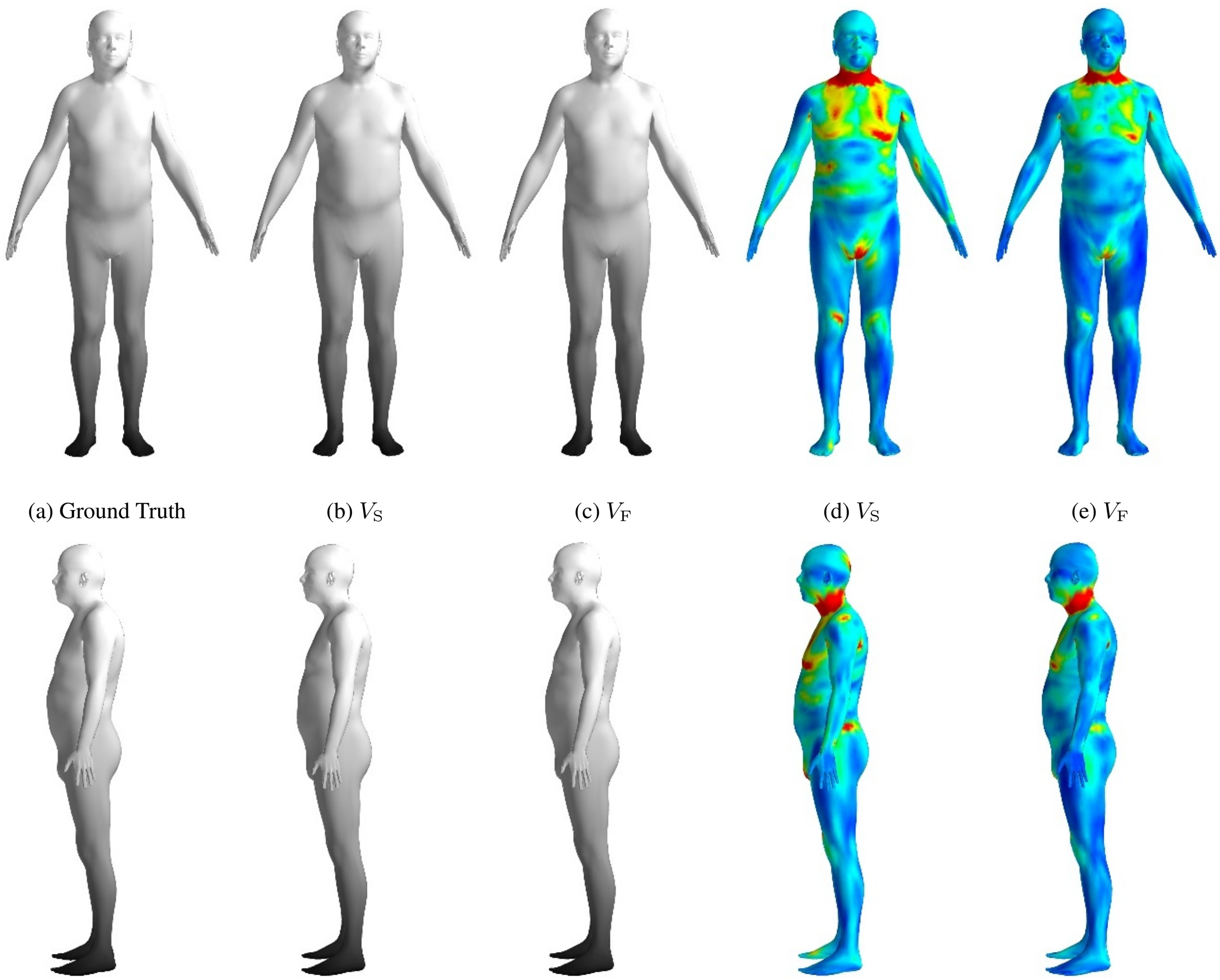
Figure 10: BMI reconstruction example.



(a) Ground Truth (b) V_{S} (c) V_{F} (d) V_{S} (e) V_{F}

(f) Ground Truth (g) V_{S} (h) V_{F} (i) V_{S} (j) V_{F}

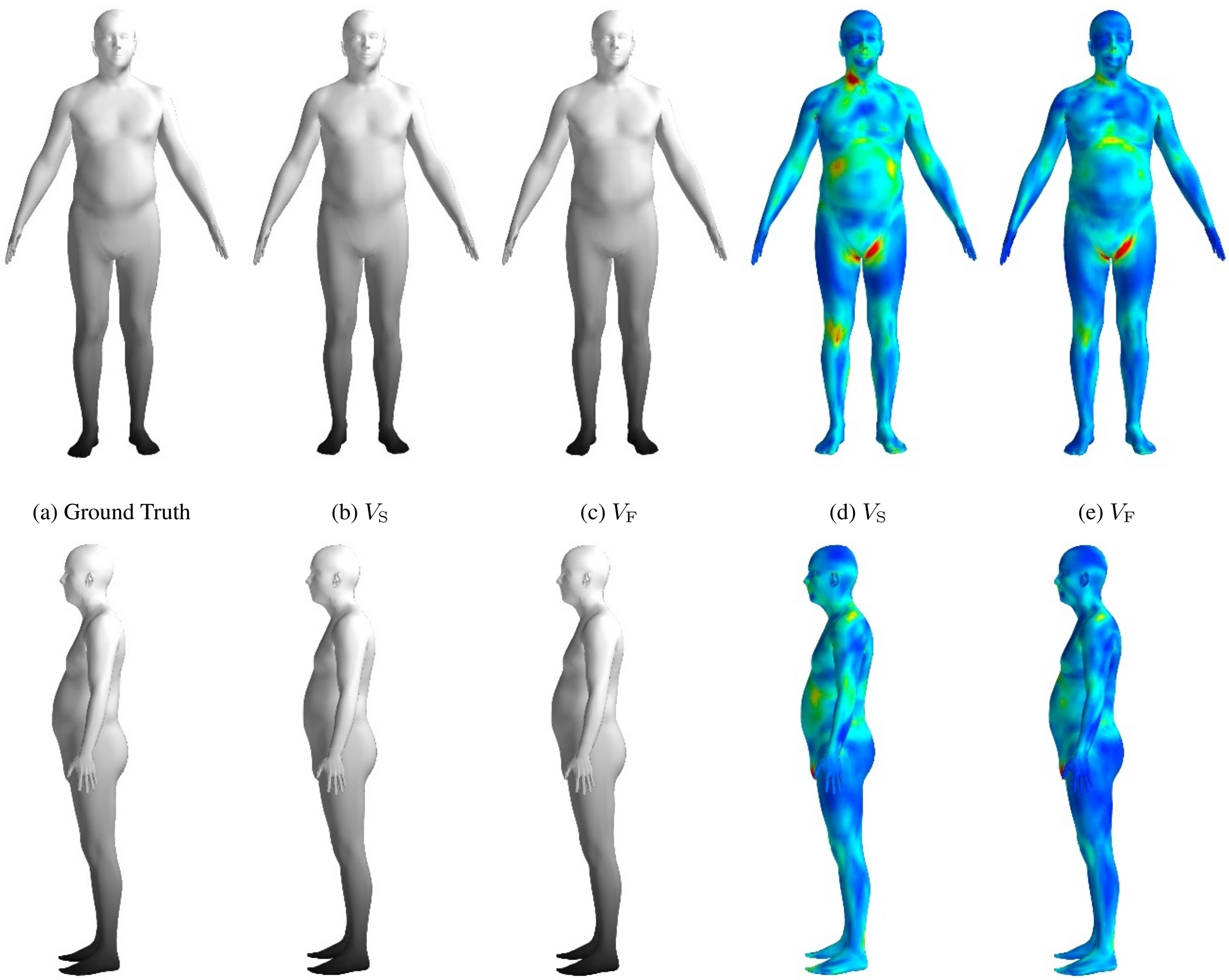
Figure 11: Gender reconstruction example.



(a) Ground Truth (b) V_{S} (c) V_{F} (d) V_{S} (e) V_{F}

(f) Ground Truth (g) V_{S} (h) V_{F} (i) V_{S} (j) V_{F}

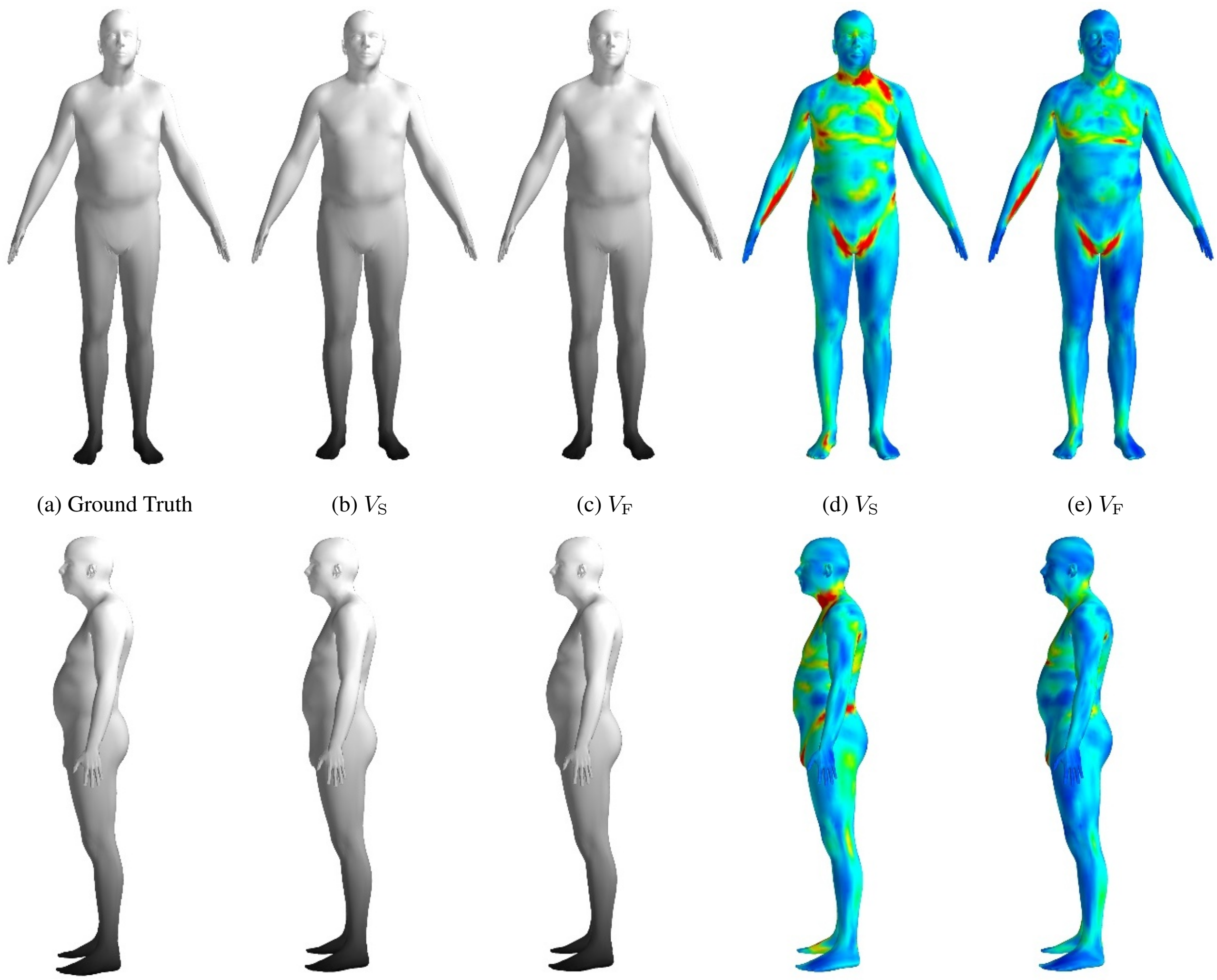
Figure 12: Gender reconstruction example.



(a) Ground Truth (b) V_{S} (c) V_{F} (d) V_{S} (e) V_{F}

(f) Ground Truth (g) V_{S} (h) V_{F} (i) V_{S} (j) V_{F}

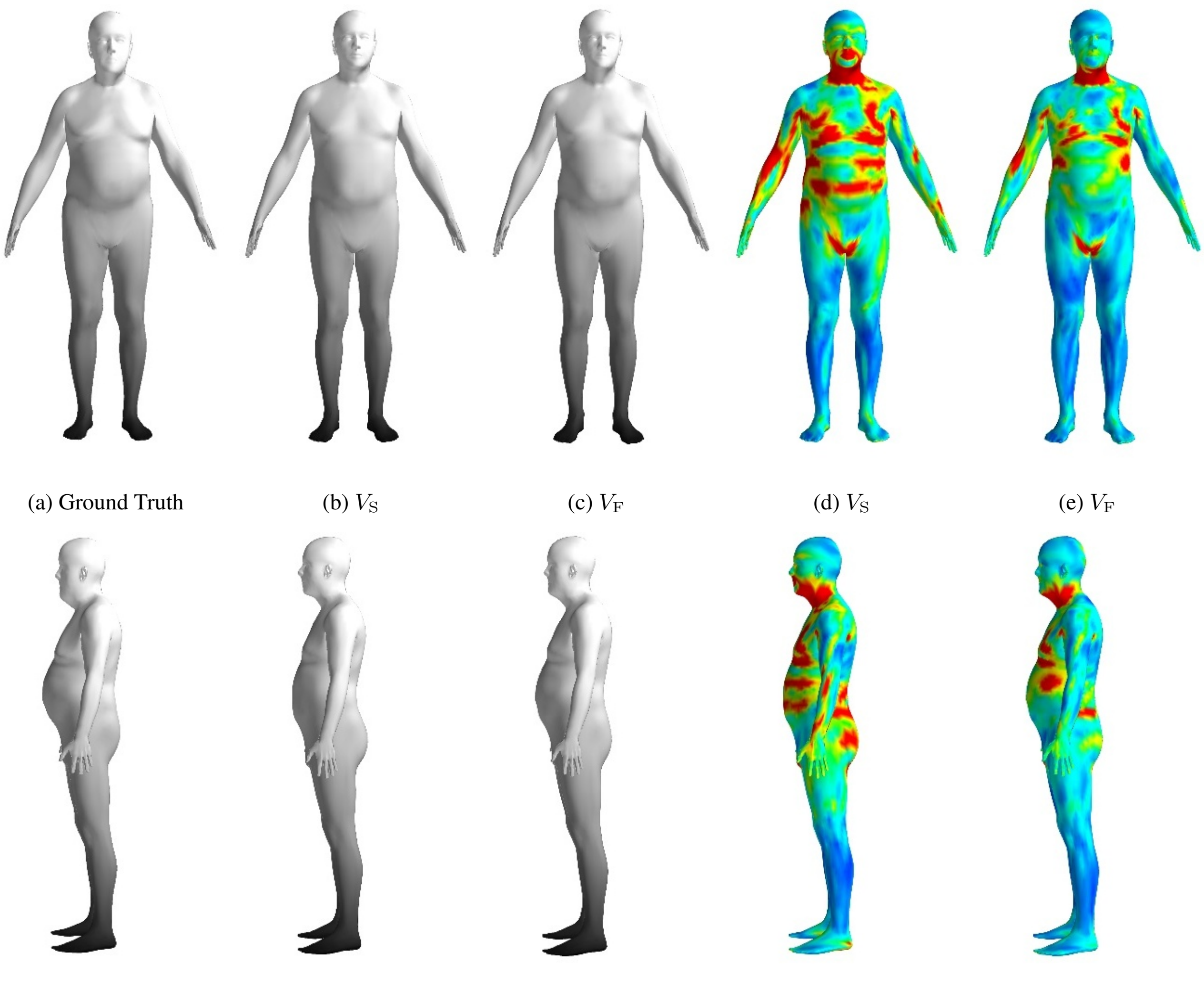
Figure 13: Gender reconstruction example.



(a) Ground Truth (b) V_{S} (c) V_{F} (d) V_{S} (e) V_{F}

(f) Ground Truth (g) V_{S} (h) V_{F} (i) V_{S} (j) V_{F}

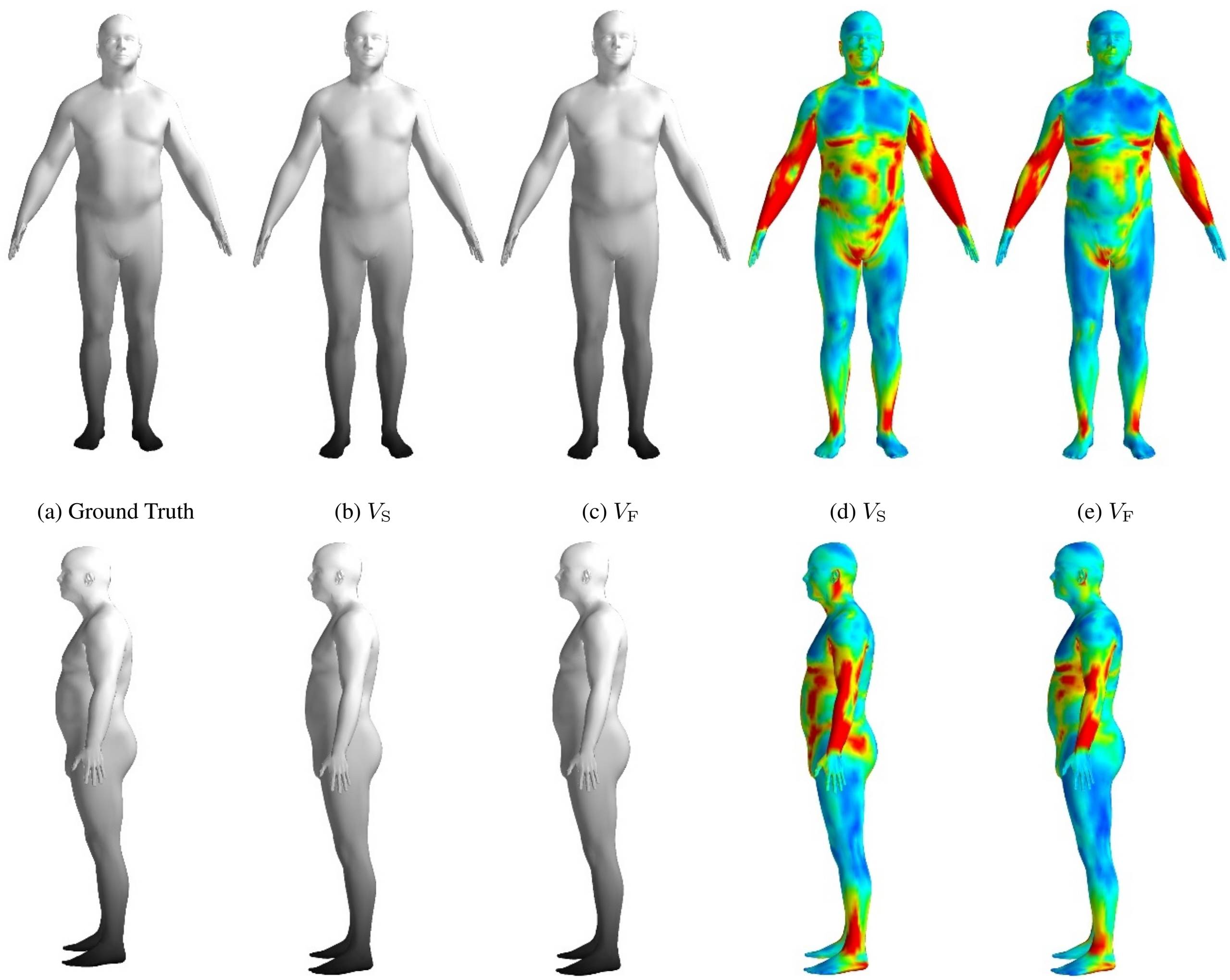
Figure 14: Gender reconstruction example.



(a) Ground Truth (b) V_{S} (c) V_{F} (d) V_{S} (e) V_{F}

(f) Ground Truth (g) V_{S} (h) V_{F} (i) V_{S} (j) V_{F}

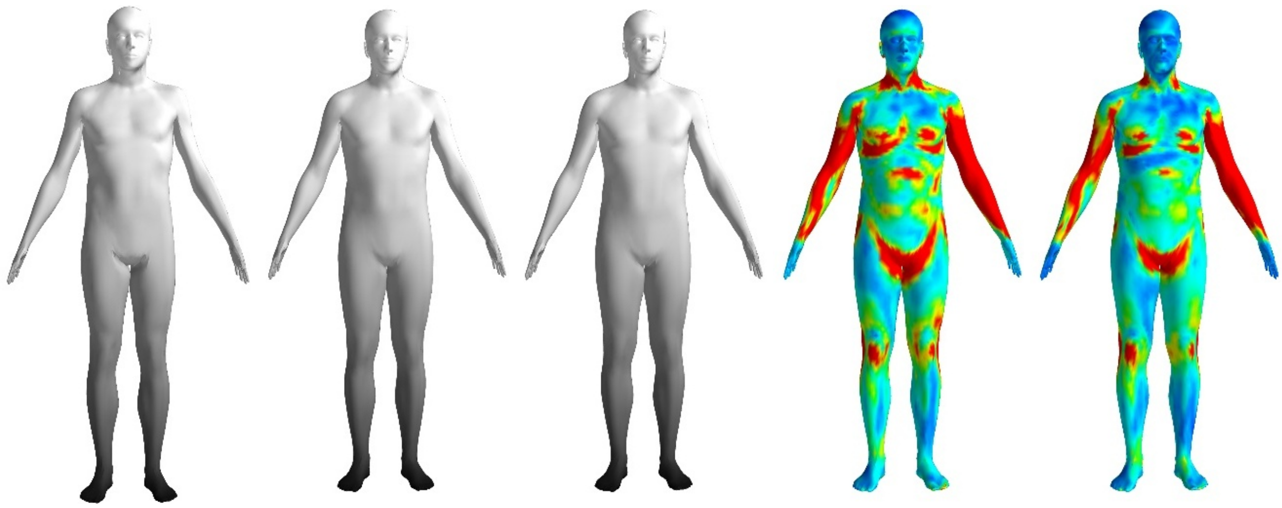
Figure 15: Gender reconstruction example.



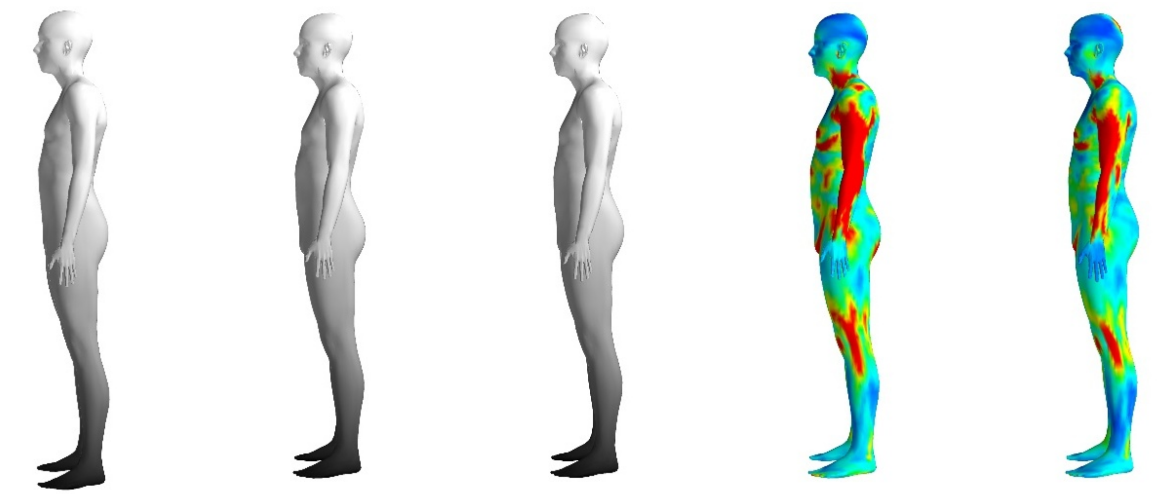
(a) Ground Truth (b) V_{S} (c) V_{F} (d) V_{S} (e) V_{F}

(f) Ground Truth (g) V_{S} (h) V_{F} (i) V_{S} (j) V_{F}

Figure 16: Gender reconstruction example.

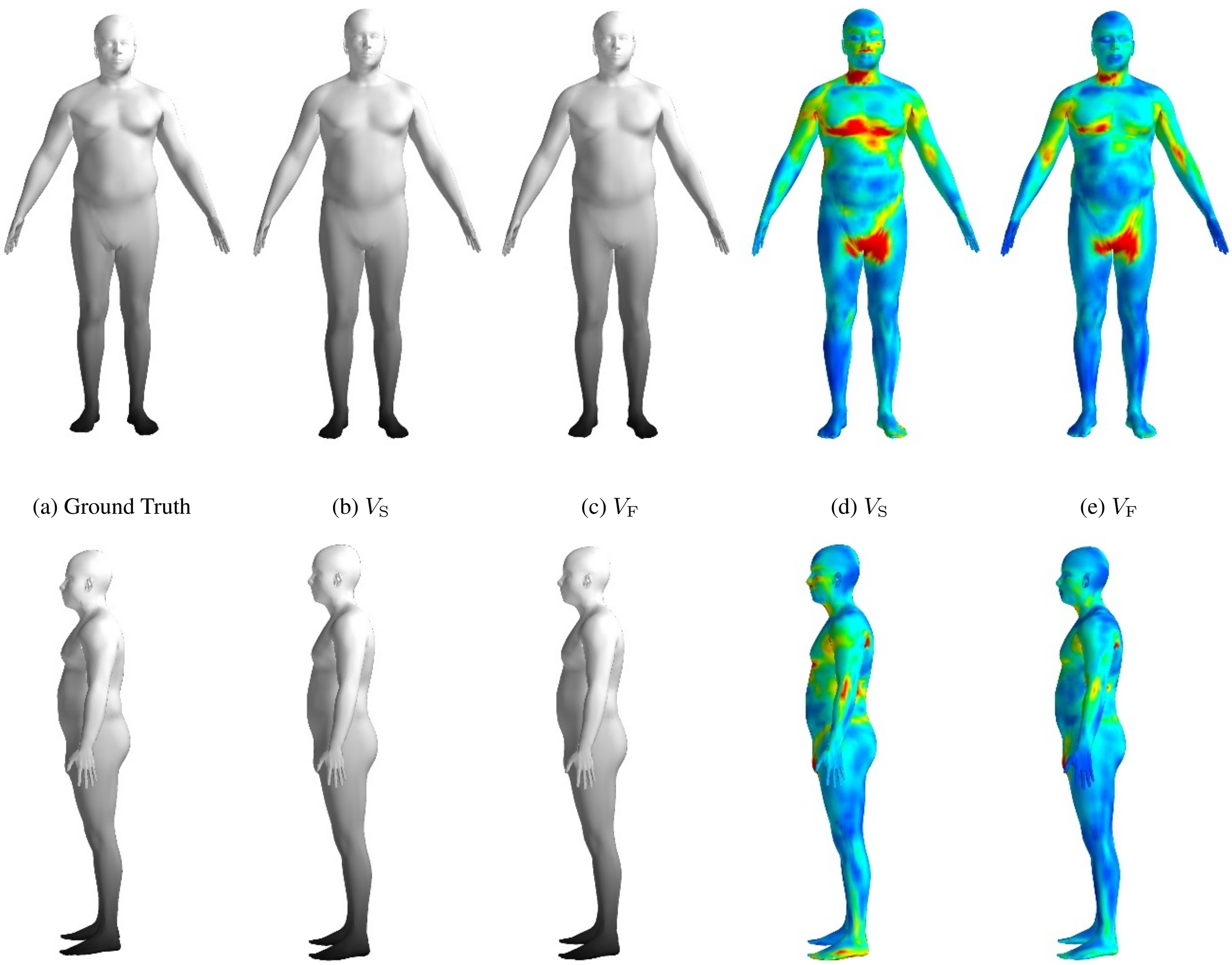


(a) Ground Truth (b) V_{S} (c) V_{F} (d) V_{S} (e) V_{F}



(f) Ground Truth (g) V_{S} (h) V_{F} (i) V_{S} (j) V_{F}

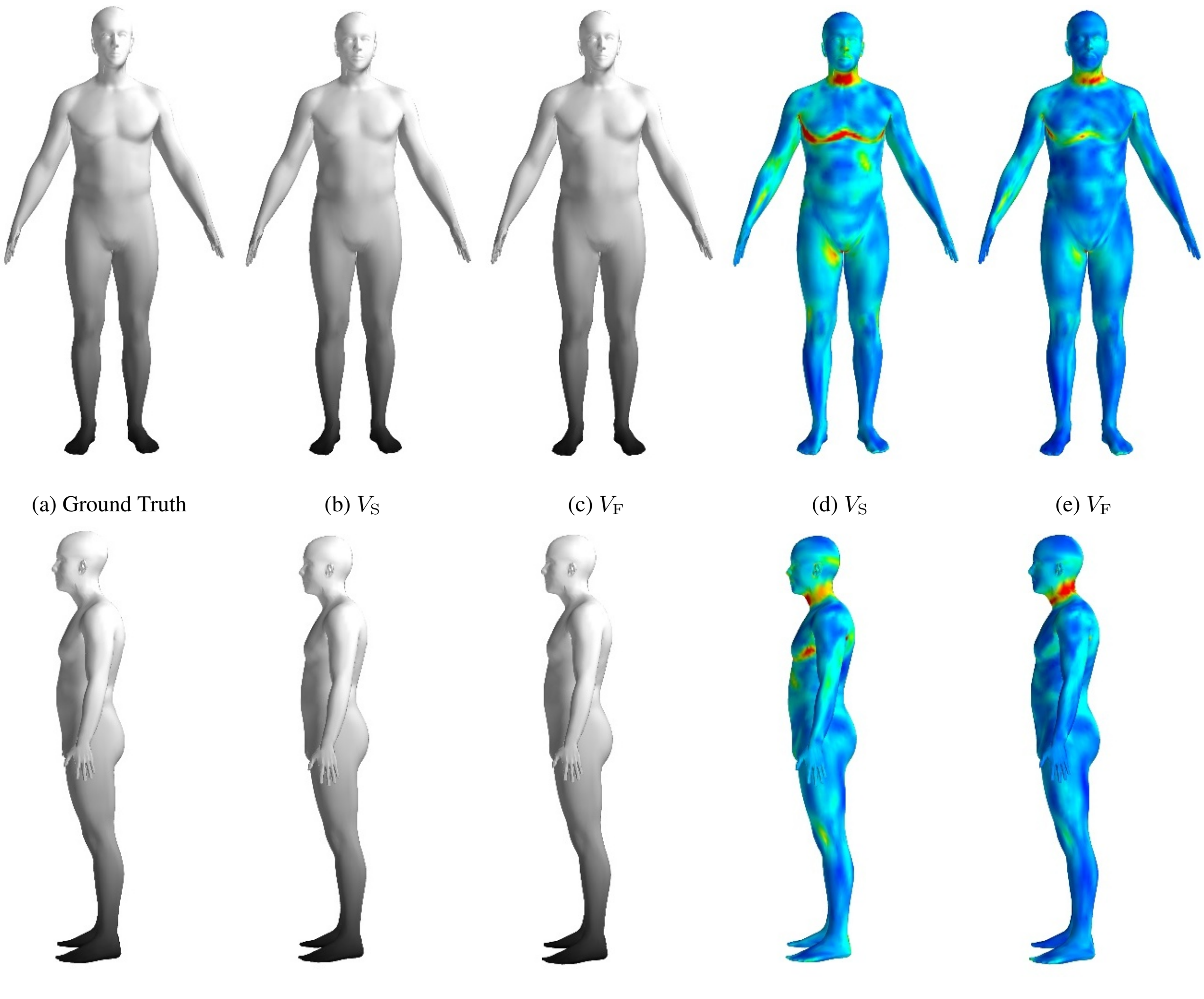
Figure 17: Gender reconstruction example.



(a) Ground Truth (b) V_{S} (c) V_{F} (d) V_{S} (e) V_{F}

(f) Ground Truth (g) V_{S} (h) V_{F} (i) V_{S} (j) V_{F}

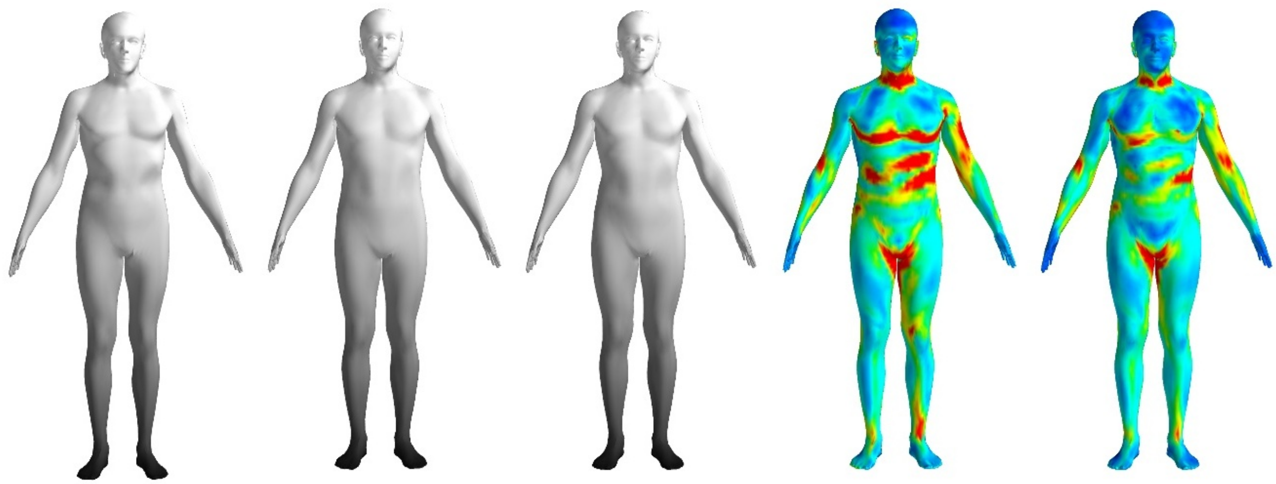
Figure 18: Gender reconstruction example.



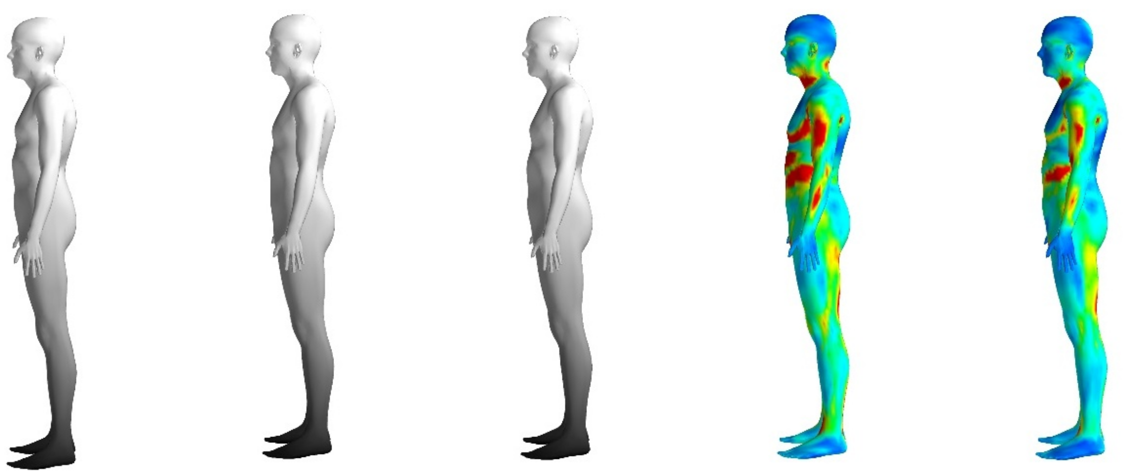
(a) Ground Truth (b) V_{S} (c) V_{F} (d) V_{S} (e) V_{F}

(f) Ground Truth (g) V_{S} (h) V_{F} (i) V_{S} (j) V_{F}

Figure 19: Gender reconstruction example.



(a) Ground Truth (b) V_{S} (c) V_{F} (d) V_{S} (e) V_{F}



(f) Ground Truth (g) V_{S} (h) V_{F} (i) V_{S} (j) V_{F}

Figure 20: Gender reconstruction example.